

Stable-Range Approach to Short Wave and Khokhlov-Zabolotskaya Equations¹

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Abstract

Short wave equations were introduced in connection with the nonlinear reflection of weak shock waves. They also relate to the modulation of a gas-fluid mixture. Khokhlov-Zabolotskaya equation are used to describe the propagation of a diffraction sound beam in a nonlinear medium. We give a new algebraic method of solving these equations by using certain finite-dimensional stable range of the nonlinear terms and obtain large families of new explicit exact solutions parameterized by several functions for them. These parameter functions enable one to find the solutions of some related practical models and boundary value problems.

1 Introduction

Khristianovich and Rizhov [8] (1958) discovered the equations of short waves in connection with the nonlinear reflection of weak shock waves. The equations are mathematically equivalent to the following equation of their potential function u for the velocity vector:

$$2u_{tx} - 2(x + u_x)u_{xx} + u_{yy} + 2ku_x = 0, \quad (1.1)$$

where k is a real constant. For convenience, we call the above equation “the short wave equation”. The symmetry group and conservation laws of (1.1) were first studied by Kucharczyk [14] (1965) and later by Khamitova [6] (1982). Bagdoev and Petrosyan [2] (1985) showed that the modulation equation of a gas-fluid mixture coincides in main orders with the corresponding short-wave equation. Roy, Roy and De [23] (1988) found a loop algebra in the Lie symmetries for the short-wave equation. Kraenkel, Manna and Merle [13] (2000) studied nonlinear short-wave propagation in ferrites and Ermakov [3] (2006) investigated short-wave interaction in film slicks.

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Khokhlov and Zabolotskaya [7] (1969) found the equation

$$2u_{tx} + (uu_x)_x - u_{yy} = 0. \quad (1.2)$$

for quasi-plane waves in nonlinear acoustics of bounded bundles. More specifically, the equation describes the propagation of a diffraction sound beam in a nonlinear medium (cf. [4], [20]). Kupershmidt [15] (1994) constructed a geometric Hamiltonian form for the Khokhlov-Zabolotskaya equation (1.2). Certain group-invariant solutions of (1.2) were found by Korsunskii [12] (1991), and by Lin and Zhang [16] (1995). The three-dimensional generalization

$$2u_{tx} + (uu_x)_x - u_{yy} - u_{zz} = 0 \quad (1.3)$$

and its symmetries were studied by Krasil'shchik, Lychagin and Vinogradov [17] (1986) and by Schwarz [25] (1987). Martinez-Moras and Ramos [18] (1993) showed that the higher dimensional classical W-algebras are the Poisson structures associated with a higher dimensional version of the Khokhlov-Zabolotskaya hierarchy. Kacdryavtsev and Sapoznikov [9] (1998) found the symmetries for a generalized Khokhlov-Zabolotskaya equation. Sanchez [24] (2007) studied long waves in ferromagnetic media via Khokhlov-Zabolotskaya equation. Morozov [19] (2008) derived two non-equivalent coverings for the modified Khokhlov-Zabolotskaya equation from Maurer-Cartan forms of its symmetry pseudo-group. Rozanova [21, 22] (2007, 2008) studied closely related Khokhlov-Zabolotskaya-Kuzentsov equation from analytic point of view. Kostin and Panasenko [11] (2008) investigated nonlinear acoustics in heterogeneous media via Khokhlov-Zabolotskaya-Kuzentsov-type equation. All the above equations are similar nonlinear algebraic partial differential equations.

Observe that the nonlinear terms in the above equations keep some finite-dimensional polynomial space in x stable. In this paper, we present a new algebraic method of solving these equations by using this stability. We obtain a family of solutions of the equation (1.1) with $k = 1/2, 2$, which blow up on a moving line $y = f(t)$. They may reflect partial phenomena of gust. Moreover, we obtain another family of smooth solutions parameterized by six smooth functions of t for any k . Similar results for the equation (1.2) are also given. Furthermore, we find a family of solutions of the equation (1.3) blowing up on a rotating and translating plane $\cos \alpha(t) y + \sin \alpha(t) z = f(t)$, which may reflect partial phenomena of sound shock, and a family of solutions parameterized by time-dependent harmonic functions in y and z , whose special cases are smooth solutions. Since our solutions contain parameter functions, they can be used to solve certain related practical models and boundary-value problems for these equations.

On the list of the Lie point symmetries of the equation (1.1) in the works of Kucharczyk [14] and of Khamitova [6] (e.g. cf. Page 301 in [5]), the most sophisticated ones are those with respect to the following vector fields:

$$X_1 = -\alpha' y \partial_x + \alpha \partial_y + \left[xy(\alpha'' + \alpha') - \frac{y^3}{3}(\alpha''' + (k+1)\alpha'' + k\alpha') \right] \partial_u, \quad (1.4)$$

$$X_2 = \beta \partial_x + [y^2(\beta'' + (k+1)\beta' + k\beta) - x(\beta' + \beta)] \partial_u, \quad (1.5)$$

where α and β are arbitrary functions of t . Among the known Lie point symmetries of the Khokhlov-Zabolotskaya equation (1.2) in the works of Vinogradov and Vorob'ev [26], and of Schwarz [25] (e.g. cf. Page 299 in [5]), the most interesting ones are those with respect to the following vector fields:

$$X_3 = \frac{1}{2}\alpha'y\partial_x + \alpha\partial_y - \frac{1}{2}\alpha''y\partial_u, \quad (1.6)$$

$$X_4 = \beta\partial_t + \frac{2\beta'x + \beta''y^2}{6}\partial_x + \frac{2}{3}\beta'y\partial_y - \frac{4\beta'u + 2\beta''x + \beta'''y^2}{6}\partial_u. \quad (1.7)$$

The symmetries of the three-dimensional Khokhlov-Zabolotskaya equation (1.3) causing our attention are those with respect to the vector fields (e.g. cf. Page 301 in [5]):

$$X_5 = 10t^2\partial_t + (4tx + 3y^2 + 3z^2)\partial_x + 12ty\partial_y + 12tz\partial_z - (4x + 16tu)\partial_u, \quad (1.7)$$

$$X_6 = \frac{1}{2}\alpha'y\partial_x + \alpha\partial_z - \frac{1}{2}\alpha''y\partial_u, \quad (1.8)$$

$$X_7 = \frac{1}{2}\beta'z\partial_x + \beta\partial_y - \frac{1}{2}\beta''z\partial_u. \quad (1.9)$$

We find that the group-invariant solutions with respect to the above vector fields X_1 - X_7 are polynomial in x . This motivates us to find more exact solutions of the equations (1.1)-(1.3) polynomial in x .

In Section 2, we solve the short-wave equation (1.1). Although the equation (1.2) can be viewed as a special case of the equation (1.3), we first solve (1.2) in Section 3 for simplicity because our approach to (1.3) involves time-dependent harmonic functions and sophisticated integrals. The exact solutions of the equation (1.3) will be given Section 4.

2 Short Wave Equation

In this section, we study solutions polynomial in x for the short wave equation (1.1). By comparing the terms of highest degree in x , we find that such a solution must be of the form:

$$u = f(t, y) + g(t, y)x + h(t, y)x^2 + \xi(t, y)x^3, \quad (2.1)$$

where $f(t, y)$, $g(t, y)$, $h(t, y)$ and $\xi(t, y)$ are suitably-differentiable functions to be determined. Note

$$u_x = g + 2hx + 3\xi x^2, \quad u_{xx} = 2h + 6\xi x, \quad (2.2)$$

$$u_{tx} = g_t + 2h_tx + 3\xi_tx^2, \quad u_{yy} = f_{yy} + g_{yy}x + h_{yy}x^2 + \xi_{yy}x^3, \quad (2.3)$$

Now (1.1) becomes

$$\begin{aligned} & 2(g_t + 2h_tx + 3\xi_tx^2) - 2(g + (2h + 1)x + 3\xi x^2)(2h + 6\xi x) \\ & + f_{yy} + g_{yy}x + h_{yy}x^2 + \xi_{yy}x^3 + 2k(g + 2hx + 3\xi x^2) = 0, \end{aligned} \quad (2.4)$$

which is equivalent to the following systems of partial differential equations:

$$\xi_{yy} = 36\xi^2, \quad (2.5)$$

$$h_{yy} = 6\xi(6h + 2 - k) - 6\xi_t, \quad (2.6)$$

$$g_{yy} = 8h^2 + 4(1 - k)h + 12\xi g - 4h_t, \quad (2.7)$$

$$f_{yy} = 4gh - 2g_t - 2kg. \quad (2.8)$$

First we observe that

$$\xi = \frac{1}{(\sqrt{6}y + \beta(t))^2} \quad (2.9)$$

is a solution of the equation (2.5) for any differentiable function β of t . Substituting (2.9) into (2.6), we get

$$h_{yy} = \frac{12\beta'(t)}{(\sqrt{6}y + \beta(t))^3} + \frac{6(6h + 2 - k)}{(\sqrt{6}y + \beta(t))^2}. \quad (2.10)$$

Denote by \mathbb{Z} the ring of integers. Write

$$h(t, y) = \sum_{i \in \mathbb{Z}} a_i(t)(\sqrt{6}y + \beta(t))^i. \quad (2.11)$$

Then

$$h_{yy} = \sum_{i \in \mathbb{Z}} 6(i+2)(i+1)a_{i+2}(t)(\sqrt{6}y + \beta(t))^i. \quad (2.12)$$

Substituting (2.11) and (2.12) into (2.10), we obtain

$$\sum_{i \in \mathbb{Z}} 6[(i+2)(i+1) - 6]a_{i+2}(t)(\sqrt{6}y + \beta(t))^i = \frac{12\beta'(t)}{(\sqrt{6}y + \beta(t))^3} + \frac{6(2 - k)}{(\sqrt{6}y + \beta(t))^2}. \quad (2.13)$$

So

$$-24a_{-1}(t) = 12\beta'(t), \quad -36a_0(t) = 6(2 - k) \quad (2.14)$$

and

$$6(i+4)(i-1)a_{i+2}(t) = 0, \quad i \neq -2, -3. \quad (2.15)$$

Thus

$$h = \frac{\alpha}{(\sqrt{6}y + \beta)^2} - \frac{\beta'}{2(\sqrt{6}y + \beta)} + \frac{k-2}{6} + \gamma(\sqrt{6}y + \beta)^3, \quad (2.16)$$

where α and γ are arbitrary differentiable functions of t .

Note

$$h_t = \frac{-2\alpha\beta'}{(\sqrt{6}y + \beta)^3} + \frac{2\alpha' + (\beta')^2}{2(\sqrt{6}y + \beta)^2} - \frac{\beta''}{2(\sqrt{6}y + \beta)} + 3\gamma\beta'(\sqrt{6}y + \beta)^2 + \gamma'(\sqrt{6}y + \beta)^3 \quad (2.17)$$

and

$$\begin{aligned} h^2 &= \frac{\alpha^2}{(\sqrt{6}y + \beta)^4} - \frac{\alpha\beta'}{(\sqrt{6}y + \beta)^3} + \frac{3(\beta')^2 + 4(k-2)\alpha}{12(\sqrt{6}y + \beta)^2} + \frac{(2-k)\beta'}{6(\sqrt{6}y + \beta)} + \frac{(k-2)^2}{36} \\ &\quad + 2\alpha\gamma(\sqrt{6}y + \beta) - \beta'\gamma(\sqrt{6}y + \beta)^2 + \frac{(k-2)\gamma}{3}(\sqrt{6}y + \beta)^3 + \gamma^2(\sqrt{6}y + \beta)^6. \end{aligned} \quad (2.18)$$

Substituting the above two equations into (2.7), we have:

$$g_{yy} - \frac{12g}{(\sqrt{6}y + \beta)^2} = \frac{8\alpha^2}{(\sqrt{6}y + \beta)^4} - \frac{4[(k+1)\alpha + 3\alpha']}{3(\sqrt{6}y + \beta)^2} + \frac{2((k+1)\beta' + 3\beta'')}{3(\sqrt{6}y + \beta)}$$

$$\begin{aligned}
& + \frac{2(k-2)(1-2k)}{9} + 16\alpha\gamma(\sqrt{6}y + \beta) - 20\beta'\gamma(\sqrt{6}y + \beta)^2 \\
& - \frac{4[(k+1)\gamma + 3\gamma']}{3}(\sqrt{6}y + \beta)^3 + 8\gamma^2(\sqrt{6}y + \beta)^6. \tag{2.19}
\end{aligned}$$

Write

$$g(t, y) = \sum_{i \in \mathbb{Z}} b_i(t)(\sqrt{6}y + \beta)^i. \tag{2.20}$$

Then

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} 6[(i+2)(i+1)-2]b_{i+2}(t)(\sqrt{6}y + \beta)^i \\
= & \frac{8\alpha^2}{(\sqrt{6}y + \beta)^4} - \frac{4[(k+1)\alpha + 3\alpha']}{3(\sqrt{6}y + \beta)^2} + \frac{2[(k+1)\beta' + 3\beta'']}{3(\sqrt{6}y + \beta)} \\
& + \frac{2(k-2)(1-2k)}{3} + 16\alpha\gamma(\sqrt{6}y + \beta) - 20\beta'\gamma(\sqrt{6}y + \beta)^2 \\
& - \frac{4[(k+1)\gamma + 3\gamma']}{3}(\sqrt{6}y + \beta)^3 + 8\gamma^2(\sqrt{6}y + \beta)^6. \tag{2.21}
\end{aligned}$$

Comparing the constant terms, we get $k = 1/2, 2$. Moreover, the coefficients of the other terms give

$$b_{-2} = \frac{\alpha^2}{3}, \quad b_0 = \frac{(k+1)\alpha + 3\alpha'}{9}, \quad b_1 = -\frac{(k+1)\beta' + 3\beta''}{18}, \tag{2.22}$$

$$b_3 = \frac{2\alpha\gamma}{3}, \quad b_4 = -\frac{\beta'\gamma}{3}, \quad b_5 = -\frac{(k+1)\gamma + 3\gamma'}{81}, \quad b_8 = \frac{2\gamma^2}{81} \tag{2.23}$$

and

$$(i+3)ib_{i+2} = 0 \quad \text{for } i \neq -4, -2, -1, 1, 2, 3, 6. \tag{2.24}$$

Therefore

$$\begin{aligned}
g = & \frac{\alpha^2}{3(\sqrt{6}y + \beta)^2} + \frac{\sigma}{\sqrt{6}y + \beta} + \frac{(k+1)\alpha + 3\alpha'}{9} - \frac{(k+1)\beta' + 3\beta''}{18}(\sqrt{6}y + \beta) \\
& + \rho(\sqrt{6}y + \beta)^2 + \frac{2\alpha\gamma}{3}(\sqrt{6}y + \beta)^3 - \frac{\beta'\gamma}{3}(\sqrt{6}y + \beta)^4 \\
& - \frac{(k+1)\gamma + 3\gamma'}{81}(\sqrt{6}y + \beta)^5 + \frac{2\gamma^2}{81}(\sqrt{6}y + \beta)^8, \tag{2.25}
\end{aligned}$$

where σ and ρ are arbitrary differentiable functions of t .

Observe that

$$\begin{aligned}
g_t = & -\frac{2\alpha^2\beta'}{3(\sqrt{6}y + \beta)^3} + \frac{(2\alpha\alpha' - 3\sigma\beta')}{3(\sqrt{6}y + \beta)^2} + \frac{\sigma'}{\sqrt{6}y + \beta} + \frac{(k+1)(2\alpha' - (\beta')^2)}{18} \\
& + \frac{2\alpha'' - \beta'\beta''}{6} + \frac{36\beta'\rho - (k+1)\beta'' - 3\beta'''}{18}(\sqrt{6}y + \beta) + (\rho' + 2\alpha\beta'\gamma)(\sqrt{6}y + \beta)^2 \\
& + \frac{2\alpha\gamma' + 2\alpha'\gamma - 4(\beta')^2\gamma}{3}(\sqrt{6}y + \beta)^3 - \frac{42\beta'\gamma' + 5(k+1)\beta'\gamma + 27\beta''\gamma}{81}(\sqrt{6}y + \beta)^4 \\
& - \frac{(k+1)\gamma' + 3\gamma''}{81}(\sqrt{6}y + \beta)^5 + \frac{16\beta'\gamma^2}{81}(\sqrt{6}y + \beta)^7 + \frac{4\gamma\gamma'}{81}(\sqrt{6}y + \beta)^8, \tag{2.26}
\end{aligned}$$

$$\begin{aligned}
gh = & \frac{\alpha^3}{3(\sqrt{6}y + \beta)^4} + \frac{6\alpha\sigma - \alpha^2\beta'}{6(\sqrt{6}y + \beta)^3} + \frac{k\alpha^2 + 2\alpha\alpha' - 3\beta'\sigma}{6(\sqrt{6}y + \beta)^2} \\
& + \frac{3(k-2)\sigma - 3\alpha\beta'' - 2(k+1)\alpha\beta' - 3\alpha'\beta'}{18(\sqrt{6}y + \beta)} + \alpha\rho + \frac{(k+1)(\beta')^2 + 3\beta'\beta''}{36} \\
& + \frac{(k-2)((k+1)\alpha + 3\alpha')}{54} + [\alpha^2\gamma - \frac{\beta'\rho}{2} - \frac{(k-2)((k+1)\beta' + 3\beta'')}{108}] (\sqrt{6}y + \beta) \\
& + \frac{(k-2)\rho + 6\gamma\sigma - 4\alpha\beta'\gamma}{6} (\sqrt{6}y + \beta)^2 + [\frac{(\beta')^2\gamma}{6} + \frac{(17k-10)\alpha\gamma - 3\alpha\gamma'}{81} \\
& + \frac{\alpha'\gamma}{3}] (\sqrt{6}y + \beta)^3 + \frac{(10-17k)\beta'\gamma + 3\beta'\gamma' - 27\beta''\gamma}{162} (\sqrt{6}y + \beta)^4 \\
& + [\gamma\rho - \frac{(k-2)((k+1)\gamma + 3\gamma')}{486}] (\sqrt{6}y + \beta)^5 + \frac{56\alpha\gamma^2}{81} (\sqrt{6}y + \beta)^6 \\
& - \frac{28\beta'\gamma^2}{81} (\sqrt{6}y + \beta)^7 + \frac{(k-2)\gamma^2}{243} (\sqrt{6}y + \beta)^8 + \frac{2\gamma^3}{81} (\sqrt{6}y + \beta)^{11}. \tag{2.27}
\end{aligned}$$

Substituting (2.25), (2.26) and (2.27) into (2.8), we obtain

$$\begin{aligned}
f_{yy} = & \frac{4\alpha^3}{3(\sqrt{6}y + \beta)^4} - \frac{2[6(k+1)\sigma + 3\alpha\beta'' + 2(k+1)\alpha\beta' + 3\alpha'\beta' + 9\sigma']}{9(\sqrt{6}y + \beta)} \\
& + \frac{12\alpha\sigma + 2\alpha^2\beta'}{3(\sqrt{6}y + \beta)^3} + 4\alpha\rho + \frac{2[\beta'\beta'' - (k+1)\alpha' - \alpha'']}{3} + \frac{2(k+1)(\beta')^2}{9} \\
& - \frac{4(k+1)^2\alpha}{27} + [4\alpha^2\gamma - 6\beta'\rho + \frac{\beta'''+(k+1)\beta''}{3} + \frac{2(k+1)^2\beta'}{27}] (\sqrt{6}y + \beta) \\
& + [4\gamma\sigma - 2\rho' - \frac{4(k+1)\rho + 20\alpha\beta'\gamma}{3}] (\sqrt{6}y + \beta)^2 + [-\frac{40((k+1)\alpha\gamma + 3\alpha\gamma')}{81} \\
& + \frac{10(\beta')^2\gamma}{3}] (\sqrt{6}y + \beta)^3 + \frac{10(k+1)\beta'\gamma + 30\beta'\gamma'}{27} (\sqrt{6}y + \beta)^4 + [4\gamma\rho \\
& + \frac{4(k+1)^2\gamma}{243} + \frac{2(k+1)\gamma' + 2\gamma''}{27}] (\sqrt{6}y + \beta)^5 + \frac{224\alpha\gamma^2}{81} (\sqrt{6}y + \beta)^6 \\
& - \frac{16\beta'\gamma^2}{9} (\sqrt{6}y + \beta)^7 - \frac{8\gamma((k+1)\gamma + 3\gamma')}{243} (\sqrt{6}y + \beta)^8 + \frac{8\gamma^3}{81} (\sqrt{6}y + \beta)^{11}. \tag{2.28}
\end{aligned}$$

Thus

$$\begin{aligned}
f = & \frac{\alpha^3}{27(\sqrt{6}y + \beta)^2} + \frac{6\alpha\sigma + \alpha^2\beta'}{18(\sqrt{6}y + \beta)} + \theta + \vartheta y + 2\alpha\rho y^2 + \frac{(k+1)(\beta')^2}{9} y^2 \\
& - \frac{6(k+1)\sigma + 3\alpha\beta'' + 2(k+1)\alpha\beta' + 3\alpha'\beta' + 9\sigma'}{27} (\sqrt{6}y + \beta) [\ln(\sqrt{6}y + \beta) - 1] \\
& + \frac{\beta'\beta'' - (k+1)\alpha' - \alpha''}{3} y^2 - \frac{2(k+1)^2\alpha}{27} y^2 + [\frac{\alpha^2\gamma}{9} - \frac{\beta'\rho}{6} + \frac{\beta'''+(k+1)\beta''}{108} \\
& + \frac{(k+1)^2\beta'}{486}] (\sqrt{6}y + \beta)^3 + [\frac{2\gamma\sigma - \rho'}{36} - \frac{(k+1)\rho + 5\alpha\beta'\gamma}{54}] (\sqrt{6}y + \beta)^4 \\
& + [\frac{(\beta')^2\gamma}{36} - \frac{(k+1)\alpha\gamma + 3\alpha\gamma'}{243}] (\sqrt{6}y + \beta)^5 + \frac{(k+1)\beta'\gamma + 3\beta'\gamma'}{486} (\sqrt{6}y + \beta)^6 \\
& + [\frac{\gamma\rho}{63} + \frac{(k+1)^2\gamma}{15309} + \frac{(k+1)\gamma' + \gamma''}{3402}] (\sqrt{6}y + \beta)^7 + \frac{2\alpha\gamma^2}{243} (\sqrt{6}y + \beta)^8 \\
& - \frac{\beta'\gamma^2}{243} (\sqrt{6}y + \beta)^9 - \frac{2\gamma((k+1)\gamma + 3\gamma')}{32805} (\sqrt{6}y + \beta)^{10} + \frac{\gamma^3}{9477} (\sqrt{6}y + \beta)^{13}, \tag{2.29}
\end{aligned}$$

where θ and ϑ are arbitrary functions of t .

Theorem 2.1. When $k = 1/2, 2$, we have the following solution of the equation (1.1) blowing up on the surface $\sqrt{6}y + \beta(t) = 0$:

$$\begin{aligned}
u = & \frac{x^3}{(\sqrt{6}y + \beta)^2} + \left[\frac{\alpha}{(\sqrt{6}y + \beta)^2} - \frac{\beta'}{2(\sqrt{6}y + \beta)} + \frac{k-2}{6} + \gamma(\sqrt{6}y + \beta)^3 \right] x^2 \\
& + \left[\frac{\alpha^2}{3(\sqrt{6}y + \beta)^2} + \frac{\sigma}{\sqrt{6}y + \beta} + \frac{(k+1)\alpha + 3\alpha'}{9} - \frac{(k+1)\beta' + 3\beta''}{18} \right] (\sqrt{6}y + \beta) \\
& + \rho(\sqrt{6}y + \beta)^2 + \frac{2\alpha\gamma}{3}(\sqrt{6}y + \beta)^3 - \frac{\beta'\gamma}{3}(\sqrt{6}y + \beta)^4 - \frac{(k+1)\gamma + 3\gamma'}{81}(\sqrt{6}y + \beta)^5 \\
& + \frac{2\gamma^2}{81}(\sqrt{6}y + \beta)^8]x + \frac{\alpha^3}{27(\sqrt{6}y + \beta)^2} + \frac{6\alpha\sigma + \alpha^2\beta'}{18(\sqrt{6}y + \beta)} + 2\alpha\rho y^2 + \frac{(k+1)(\beta')^2}{9}y^2 \\
& - \frac{6(k+1)\sigma + 3\alpha\beta'' + 2(k+1)\alpha\beta' + 3\alpha'\beta' + 9\sigma'}{27}(\sqrt{6}y + \beta)[\ln(\sqrt{6}y + \beta) - 1] \\
& + \frac{\beta'\beta'' - (k+1)\alpha' - \alpha''}{3}y^2 - \frac{2(k+1)^2\alpha}{27}y^2 + \left[\frac{\alpha^2\gamma}{9} - \frac{\beta'\rho}{6} + \frac{\beta'''' + (k+1)\beta'}{108} \right. \\
& \left. + \frac{(k+1)^2\beta'}{486} \right] (\sqrt{6}y + \beta)^3 + \left[\frac{2\gamma\sigma - \rho'}{36} - \frac{(k+1)\rho + 5\alpha\beta'\gamma}{54} \right] (\sqrt{6}y + \beta)^4 + \theta + \vartheta y \\
& + \left[\frac{(\beta')^2\gamma}{36} - \frac{(k+1)\alpha\gamma + 3\alpha\gamma'}{243} \right] (\sqrt{6}y + \beta)^5 + \frac{(k+1)\beta'\gamma + 3\beta'\gamma'}{486}(\sqrt{6}y + \beta)^6 \\
& + \left[\frac{\gamma\rho}{63} + \frac{(k+1)^2\gamma}{15309} + \frac{(k+1)\gamma' + \gamma''}{3402} \right] (\sqrt{6}y + \beta)^7 + \frac{2\alpha\gamma^2}{243}(\sqrt{6}y + \beta)^8 \\
& - \frac{\beta'\gamma^2}{243}(\sqrt{6}y + \beta)^9 - \frac{2\gamma((k+1)\gamma + 3\gamma')}{32805}(\sqrt{6}y + \beta)^{10} + \frac{\gamma^3}{9477}(\sqrt{6}y + \beta)^{13}, \tag{2.30}
\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma, \rho, \theta$ and ϑ are arbitrary functions of t , whose derivatives appeared in the above exist in a certain open set of \mathbb{R} .

When $\alpha = \gamma = \sigma = \rho = \theta = \vartheta = 0$, the above solution becomes

$$\begin{aligned}
u = & \frac{x^3}{(\sqrt{6}y + \beta)^2} + \left[\frac{k-2}{6} - \frac{\beta'}{2(\sqrt{6}y + \beta)} \right] x^2 - \frac{(k+1)\beta' + 3\beta''}{18}(\sqrt{6}y + \beta)x \\
& + \frac{(k+1)(\beta')^2}{9}y^2 + \frac{\beta'\beta''}{3}y^2 + \left[\frac{\beta'''' + (k+1)\beta''}{108} + \frac{(k+1)^2\beta'}{486} \right] (\sqrt{6}y + \beta)^3. \tag{2.31}
\end{aligned}$$

Take the trivial solution $\xi = 0$ of (2.5), which is the only solution polynomial in y . Then (2.6) and (2.7) become

$$h_{yy} = 0, \quad g_{yy} = 8h^2 + 4(1-k)h - 4h_t, \tag{2.32}$$

Thus

$$h = \alpha(t) + \beta(t)y. \tag{2.33}$$

Hence

$$g_{yy} = 4(2\alpha^2 + (1-k)\alpha - \alpha') + 4(4\alpha\beta + (1-k)\beta - \beta')y + 8\beta^2y^2. \tag{2.34}$$

So

$$g = \gamma + \sigma y + 2(2\alpha^2 + (1-k)\alpha - \alpha')y^2 + \frac{2}{3}(4\alpha\beta + (1-k)\beta - \beta')y^3 + \frac{2}{3}\beta^2y^4, \tag{2.35}$$

where γ and σ are arbitrary functions of t . Now (2.8) yields

$$\begin{aligned}
f_{yy} &= 4(\alpha + \beta y)[\gamma + \sigma y + 2(2\alpha^2 + (1-k)\alpha - \alpha')y^2 + \frac{2}{3}(4\alpha\beta + (1-k)\beta - \beta')y^3 \\
&\quad + \frac{2}{3}\beta^2 y^4] - 2[\gamma' + \sigma'y + 2(4\alpha\alpha' + (1-k)\alpha' - \alpha'')y^2 + \frac{2}{3}(4\alpha'\beta + 4\alpha\beta' \\
&\quad + (1-k)\beta' - \beta'')y^3 + \frac{4}{3}\beta\beta'y^4] - 2k[\gamma + \sigma y + 2(2\alpha^2 + (1-k)\alpha - \alpha')y^2 \\
&\quad + \frac{2}{3}(4\alpha\beta + (1-k)\beta - \beta')y^3 + \frac{2}{3}\beta^2 y^4] \\
&= 4\alpha\gamma - 2\gamma' - 2k\gamma + 2(2\alpha\sigma + 2\beta\gamma - \sigma' - k\gamma)y + 8((1-k)\alpha - \alpha')\beta y^3 \\
&\quad + 4(4\alpha^3 + 2(1-2k)\alpha^2 - 6\alpha\alpha' + k(k-1)\alpha + (2k-1)\alpha' + \alpha'' + \beta\sigma)y^2 \\
&\quad + \frac{4}{3}(20\alpha^2\beta + 2(1-3k)\alpha\beta - 6\alpha\beta' - 4\alpha'\beta + (2k-1)\beta' + \beta'' - k(1-k)\beta)y^3 \\
&\quad + \frac{4}{3}(10\alpha\beta^2 + (2-3k)\beta^2 - 4\beta\beta')y^4 + \frac{8}{3}\beta^3 y^5. \tag{2.36}
\end{aligned}$$

Therefore,

$$\begin{aligned}
f &= (2\alpha\gamma - \gamma' - k\gamma)y^2 + \frac{2\alpha\sigma + 2\beta\gamma - \sigma' - k\gamma}{3}y^3 + \frac{2((1-k)\alpha - \alpha')}{5}y^5 + \tau + \rho y \\
&\quad + \frac{1}{3}(4\alpha^3 + 2(1-2k)\alpha^2 - 6\alpha\alpha' + k(k-1)\alpha + (2k-1)\alpha' + \alpha'' + \beta\sigma)y^4 \\
&\quad + \frac{1}{15}(20\alpha^2\beta + 2(1-3k)\alpha\beta - 6\alpha\beta' - 4\alpha'\beta + (2k-1)\beta' + \beta'' - k(1-k)\beta)y^5 \\
&\quad + \frac{2}{45}(10\alpha\beta^2 + (2-3k)\beta^2 - 4\beta\beta')y^6 + \frac{4\beta^3}{63}y^7. \tag{2.37}
\end{aligned}$$

Theorem 2.2. *The following is a solution of the equation (1.1):*

$$\begin{aligned}
u &= (\alpha + \beta y)x^2 + [\gamma + \sigma y + 2(2\alpha^2 + (1-k)\alpha - \alpha')y^2 + \frac{2}{3}(4\alpha\beta + (1-k)\beta - \beta')y^3 \\
&\quad + \frac{2}{3}\beta^2 y^4]x + (2\alpha\gamma - \gamma' - k\gamma)y^2 + \frac{2\alpha\sigma + 2\beta\gamma - \sigma' - k\gamma}{3}y^3 + \frac{2((1-k)\alpha - \alpha')}{5}y^5 \\
&\quad + \tau + \rho y + \frac{1}{3}(4\alpha^3 + 2(1-2k)\alpha^2 - 6\alpha\alpha' + k(k-1)\alpha + (2k-1)\alpha' + \alpha'' + \beta\sigma)y^4 \\
&\quad + \frac{1}{15}(20\alpha^2\beta + 2(1-3k)\alpha\beta - 6\alpha\beta' - 4\alpha'\beta + (2k-1)\beta' + \beta'' - k(1-k)\beta)y^5 \\
&\quad + \frac{2}{45}(10\alpha\beta^2 + (2-3k)\beta^2 - 4\beta\beta')y^6 + \frac{4\beta^3}{63}y^7, \tag{2.38}
\end{aligned}$$

where $\alpha, \beta, \gamma, \sigma, \rho$ and τ are arbitrary functions of t , whose derivatives appeared in the above exist in a certain open set of \mathbb{R} . Moreover, any solution polynomial in x and y of (1.1) must be of the above form. The above solution is smooth (analytic) if all $\alpha, \beta, \gamma, \sigma, \rho$ and τ are smooth (analytic) functions of t .

Remark 2.3. In addition to the nonzero solution (2.9) of the equation (2.5), the other nonzero solutions are of the form

$$\xi = \wp_t(\sqrt{6}y + \beta(t)), \tag{2.39}$$

where $\wp_\iota(w)$ is the Weierstrass's elliptic function such that

$$\wp'_\iota(w)^2 = 4(\wp_\iota(w)^3 - \iota), \quad (2.40)$$

and ι is a nonzero constant and β is any function of t . When β is not a constant, the solutions of (2.6)-(2.8) are extremely complicated. If β is constant, we can take $\beta = 0$ by adjusting ι . Any solution of (2.6)-(2.8) with $h \neq 0$ is also very complicated. Thus the only simple solution of the equation (1.1) in this case is

$$u = \wp_\iota(\sqrt{6}y) x^3. \quad (2.41)$$

3 2-D Khokhlov-Zabolotskaya Equation

The solution of the equation (1.2) polynomial in x must be of the form

$$u = f(t, y) + g(t, y)x + \xi(t, y)x^2. \quad (3.1)$$

Then

$$u_x = g + 2\xi x, \quad u_{tx} = g_t + 2\xi_t x, \quad u_{yy} = f_{yy} + g_{yy}x + \xi_{yy}x^2, \quad (3.2)$$

$$(uu_x)_x = \partial_x(fg + (g^2 + 2f\xi)x + 3g\xi x^2 + 2\xi^2 x^3) = g^2 + 2f\xi + 6g\xi x + 6\xi^2 x^2. \quad (3.3)$$

Substituting them into (1.2), we get

$$2(g_t + 2\xi_t x) + g^2 + 2f\xi + 6g\xi x + 6\xi^2 - f_{yy} - g_{yy}x - \xi_{yy}x^2 = 0, \quad (3.4)$$

equivalently,

$$\xi_{yy} = 6\xi^2, \quad (3.5)$$

$$g_{yy} - 6g\xi = 4\xi_t, \quad (3.6)$$

$$f_{yy} - 2f\xi = 2g_t + g^2. \quad (3.7)$$

First we observe that

$$\xi = \frac{1}{(y + \beta(t))^2} \quad (3.8)$$

is a solution of the equation (3.5) for any differentiable function β of t . Substituting (3.8) into (3.6), we obtain

$$g_{yy} - \frac{6g}{(y + \beta(t))^2} = -\frac{8\beta'(t)}{(y + \beta(t))^3}. \quad (3.9)$$

Write

$$g(t, y) = \sum_{i \in \mathbb{Z}} a_i(t)(y + \beta(t))^i. \quad (3.10)$$

Then (3.9) becomes

$$\sum_{i \in \mathbb{Z}} [(i+2)(i+1) - 6]a_{i+2}(t)(y + \beta(t))^i = -\frac{8\beta'(t)}{(y + \beta(t))^3}. \quad (3.11)$$

Thus

$$a_{-1} = 2\beta', \quad (i+4)(i-1)a_{i+2} = 0 \quad \text{for } i \neq -3. \quad (3.12)$$

Hence

$$g = \frac{\alpha(t)}{(y + \beta(t))^2} + \frac{2\beta'(t)}{y + \beta(t)} + \gamma(t)(y + \beta(y))^3, \quad (3.13)$$

where α and γ are arbitrary differentiable functions of t .

Note

$$g_t = -\frac{2\alpha\beta'}{(y + \beta)^3} + \frac{\alpha' - 2(\beta')^2}{(y + \beta)^2} + \frac{2\beta''}{y + \beta} + 3\gamma\beta'(y + \beta)^2 + \gamma'(\sqrt{3}y + \beta)^3 \quad (3.14)$$

and

$$g^2 = \frac{\alpha^2}{(y + \beta)^4} + \frac{4\alpha\beta'}{(y + \beta)^3} + \frac{4(\beta')^2}{(y + \beta)^2} + 2\alpha\gamma(y + \beta) + 4\gamma\beta'(y + \beta)^2 + \gamma^2(y + \beta)^6. \quad (3.15)$$

Substituting the above two equations into (3.7), we have:

$$\begin{aligned} f_{yy} - \frac{2f}{(y + \beta)^2} &= \frac{\alpha^2}{(y + \beta)^4} + \frac{2\alpha'}{(y + \beta)^2} + \frac{4\beta''}{y + \beta} + 2\alpha\gamma(y + \beta) \\ &\quad + 10\gamma\beta'(y + \beta)^2 + 2\gamma'(y + \beta)^3 + \gamma^2(y + \beta)^6. \end{aligned} \quad (3.16)$$

Write

$$f(t, y) = \sum_{i \in \mathbb{Z}} b_i(t)(y + \beta)^i. \quad (3.17)$$

Then (3.16) becomes

$$\begin{aligned} \sum_{i \in \mathbb{Z}} [(i+2)(i+1) - 2]b_{i+2}(y + \beta)^i &= \frac{\alpha^2}{(y + \beta)^4} + \frac{2\alpha'}{(y + \beta)^2} + \frac{4\beta''}{y + \beta} \\ &\quad + 2\alpha\gamma(y + \beta) + 10\beta'\gamma(y + \beta)^2 + 2\gamma'(y + \beta)^3 + \gamma^2(y + \beta)^6. \end{aligned} \quad (3.18)$$

Thus

$$b_{-2} = \frac{\alpha^2}{4}, \quad b_0 = -\alpha', \quad b_1 = -2\beta'', \quad b_3 = \frac{\alpha\gamma}{2}, \quad (3.19)$$

$$b_4 = \beta'\gamma, \quad b_5 = \frac{\gamma'}{9}, \quad b_8 = \frac{\gamma^2}{54}, \quad (3.20)$$

$$(i+3)ib_{i+2} = 0 \quad \text{for } i \neq -4, -2, -1, 1, 2, 3, 6. \quad (3.21)$$

Therefore,

$$\begin{aligned} f &= \frac{\alpha^2}{4(y + \beta)^2} + \frac{\sigma}{y + \beta} - \alpha' - 2\beta''(y + \beta) + \rho(y + be)^2 \\ &\quad + \frac{\alpha\gamma}{2}(y + \beta)^3 + \beta'\gamma(y + \beta)^4 + \frac{\gamma'}{9}(y + \beta)^5 + \frac{\gamma^2}{54}(y + \beta)^8, \end{aligned} \quad (3.22)$$

where σ and ρ are arbitrary functions of t .

Theorem 3.1. *We have the following solution of the equation (1.1) blowing up on the surface $y + \beta(t) = 0$:*

$$\begin{aligned} u &= \frac{x^2}{(y + \beta)^2} + \frac{\alpha x}{(y + \beta)^2} + \frac{2\beta' x}{y + \beta} + \gamma(y + \beta)^3 x + \frac{\alpha^2}{4(y + \beta)^2} \\ &\quad + \frac{\sigma}{y + \beta} - \alpha' - 2\beta''(y + \beta) + \rho(y + be)^2 + \frac{\alpha\gamma}{2}(y + \beta)^3 \\ &\quad + \beta'\gamma(y + \beta)^4 + \frac{\gamma'}{9}(y + \beta)^5 + \frac{\gamma^2}{54}(y + \beta)^8, \end{aligned} \quad (3.23)$$

where $\alpha, \beta, \gamma, \sigma$ and ρ are arbitrary functions of t , whose derivatives appeared in the above exist in a certain open set of \mathbb{R} .

When $\alpha = \gamma = \sigma = \rho = 0$, the above solution becomes

$$u = \frac{x^2}{(y + \beta)^2} + \frac{2\beta'x}{y + \beta} - 2\beta''(y + \beta). \quad (3.24)$$

Take the trivial solution $\xi = 0$ of (3.5), which is the only solution polynomial in y . Then (3.6) and (3.7) become

$$g_{yy} = 0, \quad f_{yy} = 2g_t + g^2. \quad (3.25)$$

Thus

$$g = \alpha(t) + \beta(t)y. \quad (3.26)$$

Hence

$$f_{yy} = \alpha^2 + 2\alpha' + 2(\beta' + \alpha\beta)y + \beta^2y^2. \quad (3.27)$$

So

$$f = \gamma + \sigma y + \frac{\alpha^2 + 2\alpha'}{2}y^2 + \frac{\beta' + \alpha\beta}{3}y^3 + \frac{\beta^2}{12}y^4, \quad (3.28)$$

where γ and σ are arbitrary functions of t .

Theorem 3.2. *The following is a solution of the equation (1.2):*

$$u = (\alpha + \beta y)x + \gamma + \sigma y + \frac{\alpha^2 + 2\alpha'}{2}y^2 + \frac{\beta' + \alpha\beta}{3}y^3 + \frac{\beta^2}{12}y^4, \quad (3.29)$$

where α, β, γ and σ are arbitrary functions of t , whose derivatives appeared in the above exist in a certain open set of \mathbb{R} . Moreover, any solution polynomial in x and y of (1.2) must be of the above form. The above solution is smooth (analytic) if all α, β, γ and σ are smooth (analytic) functions of t .

Remark 3.3. In addition to the solutions in Theorems 3.1 and 3.2, the equation (1.2) has the following simple solution:

$$u = \wp_t(y) x^2, \quad (3.30)$$

where $\wp_t(w)$ is the Weierstrass's elliptic function satisfying (2.40).

4 3-D Khokhlov-Zabolotskaya Equation

By comparing the terms of highest degree, we find that a solution polynomial in x of the equation (1.3) must be of the form:

$$u = f(t, y, z) + g(t, y, z)x + \xi(t, y, z)x^2, \quad (4.1)$$

where $f(t, y, z)$, $g(t, y, z)$ and $\xi(t, y, z)$ are suitably-differentiable functions to be determined. As (3.2)-(3.7), the equation (1.3) is equivalent to:

$$\xi_{yy} + \xi_{zz} = 6\xi^2, \quad (4.2)$$

$$g_{yy} + g_{zz} - 6g\xi = 4\xi_t, \quad (4.3)$$

$$f_{yy} + f_{zz} - 2f\xi = 2g_t + g^2. \quad (4.4)$$

First we observe that

$$\xi = \frac{1}{(y \cos \alpha(t) + z \sin \alpha(t) + \beta(t))^2} \quad (4.5)$$

is a solution of the equation (4.2), where α and β are suitable differentiable functions of t . With the above ξ , (4.3) becomes

$$g_{yy} + g_{zz} - \frac{6g}{(y \cos \alpha(t) + z \sin \alpha(t) + \beta(t))^2} = -\frac{8(\alpha'(-y \sin \alpha + z \cos \alpha) + \beta')}{(y \cos \alpha + z \sin \alpha + \beta)^3}. \quad (4.6)$$

In order to solve (4.6), we change variables:

$$\zeta = \cos \alpha y + \sin \alpha z + \beta, \quad \eta = -\sin \alpha y + \cos \alpha z. \quad (4.7)$$

Then

$$\partial_y = \cos \alpha \partial_\zeta - \sin \alpha \partial_\eta, \quad \partial_z = \sin \alpha \partial_\zeta + \cos \alpha \partial_\eta. \quad (4.8)$$

Thus

$$\partial_y^2 + \partial_z^2 = (\cos \alpha \partial_\zeta - \sin \alpha \partial_\eta)^2 + (\sin \alpha \partial_\zeta + \cos \alpha \partial_\eta)^2 = \partial_\zeta^2 + \partial_\eta^2. \quad (4.9)$$

Note

$$\partial_t(\zeta) = \alpha' \eta + \beta', \quad \partial_t(\eta) = \alpha'(\beta - \zeta). \quad (4.10)$$

The equation (4.6) can be rewritten as:

$$g_{\zeta\zeta} + g_{\eta\eta} - 6\zeta^{-2}g = -8(\alpha'\eta + \beta')\zeta^{-3}. \quad (4.11)$$

In order to solve the above equation, we assume

$$g = \sum_{i \in \mathbb{Z}} a_i(t, \eta) \zeta^i. \quad (4.12)$$

Now (4.11) becomes

$$\sum_{i \in \mathbb{Z}} [((i+2)(i+1) - 6)a_{i+2} + a_{i\eta\eta}] = -8(\alpha'\eta + \beta')\zeta^{-3}, \quad (4.13)$$

which is equivalent to

$$-4a_{-1} + a_{-3\eta\eta} = -8(\alpha'\eta + \beta'), \quad (i+4)(i-1)a_{i+2} + a_{i\eta\eta} = 0 \quad \text{for } -3 \neq i \in \mathbb{Z}. \quad (4.14)$$

Hence

$$a_{-1} = \frac{1}{4}a_{-3\eta\eta} + 2(\alpha'\eta + \beta'), \quad (i+4)(i-1)a_{i+2} = -a_{i\eta\eta} \quad \text{for } -3 \neq i \in \mathbb{Z}. \quad (4.15)$$

When $i = -4$ and $i = 1$, we get $a_{-4\eta\eta} = a_{1\eta\eta} = 0$. Moreover, a_{-2} and a_3 can be any functions.

Take

$$a_3 = \sigma, \quad a_{-2} = \rho, \quad a_{-1} = 2(\alpha'\eta + \beta'), \quad (4.16)$$

$$a_1 = a_{-1-2i} = a_{-2-2i} = 0 \quad \text{for } 0 < i \in \mathbb{Z} \quad (4.17)$$

in order to avoid infinite number of negative powers of ζ in (4.12), where σ and ρ are arbitrary functions of t and η differentiable in a certain domain. By (4.15),

$$a_{3+2k} = \frac{(-1)^k \partial_\eta^{2k}(\sigma)}{\prod_{i=1}^k (2i+5)(2i)} = \frac{(-1)^k 15 \partial_\eta^{2k}(\sigma)}{(2k+5)(2k+3)(2k+1)!}, \quad (4.18)$$

$$a_{-2+2k} = \frac{(-1)^k \partial_\eta^{2k}(\rho)}{\prod_{i=1}^k (2i)(2i-5)} = \frac{(-1)^k (2k-1)(2k-3) \partial_\eta^{2k}(\rho)}{3(2k)!}. \quad (4.19)$$

Therefore,

$$\begin{aligned} g = & 2(\alpha'\eta + \beta')\zeta^{-1} + \sum_{k=0}^{\infty} (-1)^k \left[\frac{15 \partial_\eta^{2k}(\sigma) \zeta^3}{(2k+5)(2k+3)(2k+1)!} \right. \\ & \left. + \frac{(2k-1)(2k-3) \partial_\eta^{2k}(\rho) \zeta^{-2}}{3(2k)!} \right] \zeta^{2k} \end{aligned} \quad (4.20)$$

is a solution of (4.11).

By (4.9), (4.4) is equivalent to

$$f_{\zeta\zeta} + f_{\eta\eta} - 2\zeta^{-2}f = 2g_t + g^2. \quad (4.21)$$

Note

$$\begin{aligned} g_t = & 2(\alpha''\eta + \beta'' + (\alpha')^2\beta)\zeta^{-1} - 2(\alpha')^2 - 2(\alpha'\eta + \beta')^2\zeta^{-2} + \sum_{k=0}^{\infty} (-1)^k \zeta^{2k} \{ \\ & \times \left(\frac{15 \partial_\eta^{2k}(\sigma_t + \alpha'(\beta - \zeta)\sigma_\eta) \zeta^3}{(2k+5)(2k+3)(2k+1)!} + \frac{(2k-1)(2k-3) \partial_\eta^{2k}(\rho_t + \alpha'(\beta - \zeta)\rho_\eta) \zeta^{-2}}{3(2k)!} \right) \\ & + (\alpha'\eta + \beta') \left[\frac{15 \partial_\eta^{2k}(\sigma) \zeta^2}{(2k+5)(2k+1)!} + \frac{(2k-1)(2k-2)(2k-3) \partial_\eta^{2k}(\rho) \zeta^{-3}}{3(2k)!} \right] \} \}. \end{aligned} \quad (4.22)$$

For convenience of solving the equation (4.21), we denote

$$2g_t + g^2 = \sum_{i=-4}^{\infty} b_i(t, \eta) \zeta^i \quad (4.23)$$

by (4.20) and (4.22). In particular,

$$b_{-4} = \rho^2, \quad b_{-3} = 0, \quad (4.24)$$

$$b_{-2} = 2(\rho_t + \alpha'\beta\rho_\eta) + \frac{\rho_{\eta\eta}\rho}{3}, \quad (4.25)$$

$$b_{-1} = 4[\alpha''\eta + \beta'' + (\alpha')^2\beta] - 2\alpha'\rho_\eta + \frac{2}{3}(\alpha'\eta + \beta')\rho_{\eta\eta}, \quad (4.26)$$

$$b_0 = -4(\alpha')^2 + \frac{1}{3}(\rho_{t\eta\eta} + \alpha'\beta\rho_{\eta\eta\eta}) + \frac{1}{12}\partial_\eta^4(\rho)\rho + \frac{1}{36}\rho_{\eta\eta}^2. \quad (4.27)$$

Suppose that

$$f = \sum_{i \in \mathbb{Z}} c_i(t, \eta) \zeta^i \quad (4.29)$$

is a solution (4.21). Then

$$\sum_{i \in \mathbb{Z}} [(i+2)(i+1)-2)c_{i+2} + c_{i\eta\eta}] \zeta^i = \sum_{r=-4}^{\infty} b_r \zeta^r, \quad (4.30)$$

equivalently

$$(i+3)ic_{i+2} = b_i - c_{i\eta\eta}, \quad (r+3)rc_{r+2} = -c_{r\eta\eta}, \quad r < -4 \leq i. \quad (4.31)$$

By the above second equation , we take

$$c_r = 0 \quad \text{for } r < -4 \quad (4.32)$$

to avoid infinite number of negative powers of ζ in (4.29). Letting $i = -3, 0$, we get

$$b_{-3} = c_{-3\eta\eta}, \quad b_0 = c_{0\eta\eta}. \quad (4.33)$$

The first equation is naturally satisfied because $c_{-3} = -c_{-5\eta\eta}/10 = 0$. Taking $i = -2, -4$ and $r = -6$ in (4.31), we obtain

$$c_0 = \frac{1}{2}c_{-2\eta\eta} - \frac{1}{2}b_{-2}, \quad c_{-2} = \frac{1}{4}b_{-4}. \quad (4.34)$$

So

$$c_0 = \frac{1}{8}\partial_\eta^2(b_{-4}) - \frac{1}{2}b_{-2}. \quad (4.35)$$

Thus we get a constraint:

$$b_0 = \frac{1}{8}\partial_\eta^4(b_{-4}) - \frac{1}{2}\partial_\eta^2(b_{-2}), \quad (4.36)$$

equivalently,

$$\begin{aligned} & -4(\alpha')^2 + \frac{1}{3}(\rho_{t\eta\eta} + \alpha'\beta\rho_{\eta\eta\eta}) + \frac{1}{12}\partial_\eta^4(\rho)\rho + \frac{1}{36}\rho_{\eta\eta}^2 \\ &= \frac{1}{8}\partial_\eta^4(\rho^2) - \rho_{t\eta\eta} - \alpha'\beta\rho_{\eta\eta\eta} - \frac{\partial_\eta^2(\rho_{\eta\eta}\rho)}{6}. \end{aligned} \quad (4.37)$$

Thus

$$96(\rho_{t\eta\eta} + \alpha'\beta\rho_{\eta\eta\eta}) + 6\partial_\eta^4(\rho)\rho + 2\rho_{\eta\eta}^2 - 9\partial_\eta^4(\rho^2) + 12\partial_\eta^2(\rho_{\eta\eta}\rho) = 288(\alpha')^2. \quad (4.38)$$

It can be proved by considering the terms of highest degree that any solution of (4.38) polynomial in η must be of the form

$$\rho = \gamma_0(t) + \gamma_1(t)\eta + \gamma_2(t)\eta^2. \quad (4.39)$$

Then (4.38) becomes

$$6\gamma'_2 - 5\gamma_2^2 = 9(\alpha')^2. \quad (4.40)$$

So

$$\alpha' = \frac{\epsilon}{3} \sqrt{6\gamma'_2 - 5\gamma_2^2} \Rightarrow \alpha = \frac{\epsilon}{3} \int \sqrt{6\gamma'_2 - 5\gamma_2^2} dt, \quad (4.41)$$

where $\epsilon = \pm 1$. Replace β by $-\beta$ if necessary, we can take $\epsilon = 1$. Under the assumption (4.39),

$$g = \rho\zeta^{-2} + 2(\alpha'\eta + \beta')\zeta^{-1} + \frac{\gamma_2}{6} + \sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k}(\sigma)\zeta^{3+2k}}{(2k+5)(2k+3)(2k+1)!} \quad (4.42)$$

and

$$b_{-2} = 2(\rho_t + \alpha'\beta\rho_{\eta}) + \frac{2}{3}\gamma_2\rho, \quad (4.43)$$

$$b_{-1} = 4[\alpha''\eta + \beta'' + (\alpha')^2\beta] - 2\alpha'\rho_{\eta} + \frac{4}{3}(\alpha'\eta + \beta')\gamma_2, \quad (4.44)$$

$$b_0 = -4(\alpha')^2 + \frac{2}{3}\gamma'_2 + \frac{\gamma_2^2}{9}. \quad (4.45)$$

Denote

$$\Psi_{\langle\beta,\rho,\sigma\rangle}(t, \eta, \zeta) = \sum_{i=1}^{\infty} b_i \zeta^i. \quad (4.46)$$

For any real function $F(t, \eta)$ analytic at $\eta = \eta_0$, we define

$$F(t, \eta_0 + \sqrt{-1}\zeta) = \sum_{r=0}^{\infty} \frac{\partial_{\eta}^r(F)(t, \eta_0)}{r!} (\sqrt{-1}\zeta)^r. \quad (4.47)$$

Note

$$\begin{aligned} & \sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k}(\sigma)\zeta^{3+2k}}{(2k+5)(2k+3)(2k+1)!} = 15\zeta^2 \int_0^{\zeta} \left(\sum_{k=0}^{\infty} (-1)^k \frac{\partial_{\eta}^{2k}(\sigma)\tau_1^{2k}}{(2k+5)(2k+3)(2k)!} \right) d\tau_1 \\ &= 15 \int_0^{\zeta} \tau_2 \int_0^{\tau_2} \left(\sum_{k=0}^{\infty} (-1)^k \frac{\partial_{\eta}^{2k}(\sigma)\tau_1^{2k}}{(2k+5)(2k)!} \right) d\tau_1 d\tau_2 \\ &= 15\zeta^{-2} \int_0^{\zeta} \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} \left(\sum_{k=0}^{\infty} (-1)^k \frac{\partial_{\eta}^{2k}(\sigma)\tau_1^{2k}}{(2k)!} \right) d\tau_1 d\tau_2 d\tau_3 \\ &= \frac{15}{2}\zeta^{-2} \int_0^{\zeta} \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3, \end{aligned} \quad (4.48)$$

$$\begin{aligned} & \partial_t \left[\sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k}(\sigma)\zeta^{3+2k}}{(2k+5)(2k+3)(2k+1)!} \right] \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k}(\sigma_t)\zeta^{3+2k}}{(2k+5)(2k+3)(2k+1)!} + \alpha'\beta \sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k+1}(\sigma)\zeta^{3+2k}}{(2k+5)(2k+3)(2k+1)!} \\ &\quad - \alpha' \sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k+1}(\sigma)\zeta^{4+2k}}{(2k+5)(2k+3)(2k+1)!} + (\alpha'\eta + \beta') \sum_{k=0}^{\infty} (-1)^k \frac{15\partial_{\eta}^{2k}(\sigma)\zeta^{2+2k}}{(2k+5)(2k+1)!} \\ &= \frac{15}{2}\zeta^{-2} \int_0^{\zeta} \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma_t(t, \eta + \sqrt{-1}\tau_1) + \sigma_t(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3 \\ &\quad + \frac{15\alpha'(\zeta - \beta)}{2\zeta^2} \sqrt{-1}\zeta^{-2} \int_0^{\zeta} \tau_2 \int_0^{\tau_2} \tau_1 [\sigma(t, \eta + \sqrt{-1}\tau_1) - \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 \\ &\quad + \frac{15}{2}(\alpha'\eta + \beta')\zeta^{-3} \int_0^{\zeta} \tau_2^3 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2. \end{aligned} \quad (4.49)$$

Hence

$$\begin{aligned} g &= \rho\zeta^{-2} + 2(\alpha'\eta + \beta')\zeta^{-1} + \frac{\gamma_2}{6} + \frac{15}{2}\zeta^{-2} \\ &\times \int_0^\zeta \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3, \end{aligned} \quad (4.50)$$

by (4.42) and (4.48). According to (4.23) and (4.46), we have

$$\begin{aligned} \Psi_{\langle\beta,\rho,\sigma\rangle}(t, \eta, \zeta) &= \\ \frac{225}{4}\zeta^{-4} &\left(\int_0^\zeta \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3 \right)^2 \\ + 15\zeta^{-2} &\int_0^\zeta \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma_t(t, \eta + \sqrt{-1}\tau_1) + \sigma_t(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3 \\ + \frac{15\alpha'(\zeta - \beta)}{\zeta^2} \sqrt{-1} &\int_0^\zeta \tau_2 \int_0^{\tau_2} \tau_1 [\sigma(t, \eta + \sqrt{-1}\tau_1) - \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 \\ + 15(\alpha'\eta + \beta')\zeta^{-3} &\int_0^\zeta \tau_2^3 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 \\ + 15 &\left(\int_0^\zeta \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3 \right) \\ \times \zeta^{-2} &\left(\rho\zeta^{-2} + (\alpha'\eta + \beta')\zeta^{-1} + \frac{\gamma_2}{6} \right). \end{aligned} \quad (4.51)$$

Now

$$c_{-2} = \frac{\rho^2}{4} \quad (4.52)$$

by (4.24) and (4.34). According to (4.31) with $i = -3, 0$, c_{-1} and c_2 can be arbitrary. For convenience, we redenote

$$c_{-1} = \kappa(t, \eta), \quad c_2 = \omega(t, \eta). \quad (4.53)$$

Moreover, (4.24), (4.35) and (4.43) imply

$$c_0 = \frac{\rho_\eta^2}{4} - \rho_t - \alpha'\beta\rho_\eta + \frac{\gamma_2\rho}{6}. \quad (4.54)$$

Furthermore, (4.31) and (4.44) yield

$$c_1 = \frac{\kappa_{\eta\eta}}{2} - 2(\alpha''\eta + \beta'' + (\alpha')^2\beta) + \alpha'\rho_\eta - \frac{2}{3}(\alpha'\eta + \beta')\gamma_2. \quad (4.55)$$

In addition, (4.31) and (4.53) gave

$$c_{2k+3} = \frac{(-1)^{k+1}\partial_\eta^{2k+4}(\kappa)}{2(k+2)(2k+2)!} + \sum_{i=0}^k \frac{(-1)^{k-i}(i+1)(2i)!}{(k+2)(2k+2)!} \partial_\eta^{2(k-i)}(b_{2i+1}), \quad (4.56)$$

$$c_{2k+4} = \frac{(-1)^{k+1}3\partial_\eta^{2k+2}(\omega)}{(2k+5)(2k+3)!} + \sum_{i=0}^k \frac{(-1)^{k-i}(2i+3)(2i+1)!}{(2k+5)(2k+3)!} \partial_\eta^{2(k-i)}(b_{2i+2}) \quad (4.57)$$

for $0 \leq k \in \mathbb{Z}$.

Set

$$\begin{aligned}
\Phi_{\langle \beta, \rho, \sigma, \kappa, \omega \rangle}(t, \eta, \zeta) &= \kappa \zeta^{-1} + \frac{\kappa \eta \zeta}{2} + \omega \zeta^2 + \sum_{i=3}^{\infty} c_i \zeta^i \\
&= -\zeta \partial_{\zeta} \zeta^{-1} \left[\sum_{k=0}^{\infty} (-1)^k \frac{\partial_{\eta}^{2k}(\kappa) \zeta^{2k}}{(2k)!} \right] + \zeta^2 \sum_{k=0}^{\infty} (-1)^k \frac{3 \partial_{\eta}^{2k}(\omega) \zeta^{2k}}{(2k+3)(2k+1)!} \\
&\quad + \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{(-1)^{k-i}(i+1)(2i)!}{(k+2)(2k+2)!} \partial_{\eta}^{2(k-i)}(b_{2i+1}) \zeta^{2k+3} \\
&\quad + \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{(-1)^{k-i}(2i+3)(2i+1)!}{(2k+5)(2k+3)!} \partial_{\eta}^{2(k-i)}(b_{2i+2}) \zeta^{2k+4}. \tag{4.58}
\end{aligned}$$

Note

$$\begin{aligned}
&\zeta^2 \sum_{k=0}^{\infty} (-1)^k \frac{3 \partial_{\eta}^{2k}(\omega) \zeta^{2k}}{(2k+3)(2k+1)!} \\
&= \frac{3}{2} \zeta^{-1} \int_0^{\zeta} \tau_2 \int_0^{\tau_2} [\omega(t, \eta + \sqrt{-1}\tau_1) + \omega(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2. \tag{4.59}
\end{aligned}$$

Moreover,

$$\Psi_{\langle \beta, \rho, \sigma \rangle}(t, \eta, 0) = 0, \quad b_i = \frac{\partial_{\zeta}^i(\Psi_{\langle \beta, \rho, \sigma \rangle})(t, \eta, 0)}{i!} \quad \text{for } 0 < i \in \mathbb{Z}. \tag{4.60}$$

Thus

$$\begin{aligned}
\Phi_{\langle \beta, \rho, \sigma, \kappa, \omega \rangle}(t, \eta, \zeta) &= \frac{3}{2} \zeta^{-1} \int_0^{\zeta} \tau_2 \int_0^{\tau_2} [\omega(t, \eta + \sqrt{-1}\tau_1) + \omega(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 \\
&\quad - \frac{1}{2} \zeta \partial_{\zeta} \zeta^{-1} [\kappa(t, \eta + \sqrt{-1}\tau_1) + \kappa(t, \eta - \sqrt{-1}\tau_1)] \\
&\quad + \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{(-1)^{k-i}(i+1) \partial_{\eta}^{2(k-i)} \partial_{\zeta}^{2i+1}(\Psi_{\langle \beta, \rho, \sigma \rangle})(t, \eta, 0)}{(2i+1)(k+2)(2k+2)!} \zeta^{2k+3} \\
&\quad + \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{(-1)^{k-i}(2i+3) \partial_{\eta}^{2(k-i)} \partial_{\zeta}^{2i+2}(\Psi_{\langle \beta, \rho, \sigma \rangle})(t, \eta, 0)}{(2i+2)(2k+5)(2k+3)!} \zeta^{2k+4}, \tag{3.61}
\end{aligned}$$

in which the summations are finite if $\sigma(t, \eta)$ is polynomial in η . According to (4.52)-(4.58) and (4.61),

$$\begin{aligned}
f &= \Phi_{\langle \beta, \rho, \sigma, \kappa, \omega \rangle}(t, \eta, \zeta) + \frac{\rho^2}{4} \zeta^{-2} + \frac{\rho_{\eta}^2}{4} - \rho_t - \alpha' \beta \rho_{\eta} + \frac{\gamma_2 \rho}{6} \\
&\quad - [2(\alpha'' \eta + \beta'' + (\alpha')^2 \beta) - \alpha' \rho_{\eta} + \frac{2}{3}(\alpha' \eta + \beta') \gamma_2] \zeta. \tag{4.62}
\end{aligned}$$

Theorem 4.1. *In terms of the notions in (4.7), we have the following solution of the*

equation (1.3) blowing up on the hypersurface $\cos \alpha(t) y + \sin \alpha(t) z + \beta(t) = 0$ ($\zeta = 0$):

$$\begin{aligned} u &= x^2 \zeta^{-2} + [\rho \zeta^{-2} + 2(\alpha' \eta + \beta') \zeta^{-1} + \frac{\gamma_2}{6} + \frac{15}{2} \zeta^{-2} \\ &\quad \times \int_0^\zeta \tau_3 \int_0^{\tau_3} \tau_2 \int_0^{\tau_2} [\sigma(t, \eta + \sqrt{-1}\tau_1) + \sigma(t, \eta - \sqrt{-1}\tau_1)] d\tau_1 d\tau_2 d\tau_3] x \\ &\quad + \Phi_{\langle \beta, \rho, \sigma, \kappa, \omega \rangle}(t, \eta, \zeta) + \frac{\rho^2}{4} \zeta^{-2} + \frac{\rho_\eta^2}{4} - \rho_t - \alpha' \beta \rho_\eta + \frac{\gamma_2 \rho}{6} \\ &\quad - [2(\alpha'' \eta + \beta'' + (\alpha')^2 \beta) - \alpha' \rho_\eta + \frac{2}{3}(\alpha' \eta + \beta') \gamma_2] \zeta, \end{aligned} \quad (4.63)$$

where the involved parametric functions ρ is given in (4.39), α is given in (4.41) and β is any function of t . Moreover, σ , κ , ω are real functions in real variable t and η , and $\Phi_{\langle \beta, \rho, \sigma, \kappa, \omega \rangle}(t, \eta, \zeta)$ is given in (4.61) via (4.51).

When $\sigma = \kappa = \omega = 0$, the above solution becomes:

$$\begin{aligned} u &= x^2 \zeta^{-2} + [\rho \zeta^{-2} + 2(\alpha' \eta + \beta') \zeta^{-1} + \frac{\gamma_2}{6}] x + \frac{\rho^2}{4} \zeta^{-2} + \frac{\rho_\eta^2}{4} - \rho_t \\ &\quad - \alpha' \beta \rho_\eta + \frac{\gamma_2 \rho}{6} - [2(\alpha'' \eta + \beta'' + (\alpha')^2 \beta) - \alpha' \rho_\eta + \frac{2}{3}(\alpha' \eta + \beta') \gamma_2] \zeta, \end{aligned} \quad (4.64)$$

Next we consider $\xi = 0$, which is the only solution polynomial in y and z of (4.2). In this case, (4.3) and (4.4) becomes:

$$g_{yy} + g_{zz} = 0, \quad f_{yy} + f_{zz} = 2g_t + g^2. \quad (4.65)$$

The above first equation is classical two-dimensional Laplace equation, whose solutions are called *harmonic functions*. In order to find simpler expressions of the solutions of the above equations, we introduce a new notion. A complex function

$$G(\mu) \text{ is called } \textit{bar-homomorphic} \text{ if } \overline{G(\mu)} = G(\overline{\mu}). \quad (4.66)$$

For instance, trigonometric functions, polynomials with real coefficients and elliptic functions with bar-invariant periods are bar-homomorphic functions. The extended function $F(t, \mu)$ in (4.47) is bar-homomorphic in μ .

As (4.20), it can be proved by power series that the general solution of the first equation in (4.65) is:

$$g = (\sigma + \sqrt{-1}\rho)(t, y + \sqrt{-1}z) + (\sigma - \sqrt{-1}\rho)(t, y - \sqrt{-1}z), \quad (4.67)$$

where $\sigma(t, \mu)$ and $\rho(t, \mu)$ are complex functions in real variable t and bar-homomorphic in complex variable μ . Set

$$w = y + \sqrt{-1}z, \quad \overline{w} = y - \sqrt{-1}z. \quad (4.68)$$

Then the Laplace operator

$$\partial_y^2 + \partial_z^2 = 4\partial_w \partial_{\overline{w}}. \quad (4.69)$$

The second equation in (4.65) is equivalent to:

$$\begin{aligned}\partial_w \partial_{\bar{w}}(f) &= \frac{g_t}{2} + \frac{g^2}{4} = \frac{1}{2}(\sigma_t + \sqrt{-1}\rho_t)(t, w) + (\sigma_t - \sqrt{-1}\rho_t)(t, \bar{w}) \\ &\quad + \frac{1}{4}[(\sigma + \sqrt{-1}\rho)(t, w) + (\sigma - \sqrt{-1}\rho)(t, \bar{w})]^2.\end{aligned}\tag{4.70}$$

Hence the general solution of the second equation in (4.65) is:

$$\begin{aligned}f &= \int_{\bar{w}_1}^{\bar{w}} \int_{w_1}^w \left\{ \frac{1}{2}[(\sigma_t + \sqrt{-1}\rho_t)(t, \mu_1) + (\sigma_t - \sqrt{-1}\rho_t)(t, \bar{\mu}_1)] \right. \\ &\quad \left. + \frac{1}{4}[(\sigma + \sqrt{-1}\rho)(t, \mu_1) + (\sigma - \sqrt{-1}\rho)(t, \bar{\mu}_1)]^2 \right\} d\mu_1 d\bar{\mu}_1 \\ &\quad + (\kappa + \sqrt{-1}\omega)(t, w) + (\kappa - \sqrt{-1}\omega)(t, \bar{w}),\end{aligned}\tag{4.71}$$

where $\kappa(t, \mu)$ and $\omega(t, \mu)$ are complex functions in real variable t and bar-homomorphic in complex variable μ , and w_1 is a complex constant.

Theorem 4.3. *In terms of the notions in (4.67), the following is a solution polynomial in x of the equation (1.3):*

$$\begin{aligned}u &= [(\sigma + \sqrt{-1}\rho)(t, w) + (\sigma - \sqrt{-1}\rho)(t, \bar{w})]x + \int_{\bar{w}_1}^{\bar{w}} \int_{w_1}^w \left\{ \frac{1}{2}[(\sigma_t + \sqrt{-1}\rho_t)(t, \mu_1) \right. \\ &\quad \left. + (\sigma_t - \sqrt{-1}\rho_t)(t, \bar{\mu}_1)] + \frac{1}{4}[(\sigma + \sqrt{-1}\rho)(t, \mu_1) + (\sigma - \sqrt{-1}\rho)(t, \bar{\mu}_1)]^2 \right\} d\mu_1 d\bar{\mu}_1 \\ &\quad + (\kappa + \sqrt{-1}\omega)(t, w) + (\kappa - \sqrt{-1}\omega)(t, \bar{w}),\end{aligned}\tag{4.72}$$

where $\sigma(t, \mu)$, $\rho(t, \mu)$, $\kappa(t, \mu)$ and $\omega(t, \mu)$ are complex functions in real variable t and bar-homomorphic in complex variable μ (cf. (4.66)). Moreover, the above solution is smooth (analytic) if all σ , ρ , κ and ω are smooth (analytic) functions. In particular, any solution of the equation (1.3) polynomial in x, y, z must be of the form (4.72) in which σ , ρ , κ and ω are polynomial in μ .

Remark 4.4. In addition to the solutions in Theorems 4.1 and 4.2, the equation (1.3) has the following simple solution:

$$u = \wp_t(ay + bz)x^2,\tag{4.73}$$

where $\wp_t(w)$ is the Weierstrass's elliptic function and a, b are real constants such that $a^2 + b^2 = 1$.

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